*J. Fluid Mech.* (2002), *vol.* 452, *pp.* 337–359. © 2002 Cambridge University Press
DOI: 10.1017/S0022112001006760 Printed in the United Kingdom


# On stationary solutions for free quasi-parallel mixing layers with a longitudinal magnetic field

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(Received 17 January 2000 and in revised form 10 August 2001)

The paper is devoted to the theoretical investigation of the possible existence of stationary mixing layers and of their structure in nearly perfectly conducting, nearly inviscid fluids with a longitudinal magnetic field. A system of two equations is used, which generalizes the well-known Blasius equation (for flow around a semi-infinite plate) to the case under consideration. The system depends on the magnetic Prandtl number, $P_m = \nu/\nu_m$, where $\nu$ and $\nu_m$ are the usual and the magnetic viscosities, respectively.

For the existence of stationary flows the ratio between the flow velocity $v_x$ and the Alfvén velocity $c_A = H_x/(4\pi\rho)^{1/2}$ ($\rho$ being the fluid density) plays a critical role. Super-Alfvén ($v_x > c_A$) flows are possible at any value of $P_m$ and for any values of $v_x$ and $H_x$ on the layer boundaries. Sub-Alfvén ($v_x < c_A$) stationary flows are impossible at any value of $P_m$ and for any values of the differences in $v_x$ and $H_x$ across the layer, except for two cases: $P_m = 0$ and $P_m = 1$. When $P_m = 0$, i.e. when the fluid is strictly inviscid, $\nu = 0$, flow is possible in both the super- and sub-Alfvén regimes; however, the magnetic field must be uniform, $H_x = \text{const}$, $H_y = 0$ in this case. For $P_m = 1$ both flow regimes are also possible; however, the sub-Alfvén flow is possible only for a definite relationship between the magnetic field and velocity differences: $\Delta H_x = -\Delta v_x$ (in corresponding units). For the case where the relative differences in $v_x$ and $H_x$ across the layer are small, $\Delta v_x \ll \bar{v}_x$, $\Delta H_x \ll \bar{H}_x$, solutions are obtained in explicit form for arbitrary $P_m$ (here $\bar{v}_x$ and $\bar{H}_x$ are averaged over the layer). For the specific case $P_m = 1$, exact analytical solutions of basic system are found and studied in detail.

## 1. Introduction

On numerous occasions the model of a plane-parallel inviscid free shear flow serves as a good approximation of real free flows with large Reynolds numbers. Such a model has been successfully employed as the mean flow in theoretical studies on the development of small perturbations, and provides answers to many fundamental questions in hydrodynamic stability theory (see, for example, Drazin & Howard 1966).

However, there are situations where such a model is not sufficient, and proper allowance must be made for viscosity. Here, we are dealing with problems in which a substantial role is played by a so-called critical layer (CL), a narrow region on the flow profile where the phase velocity of the perturbation wave coincides with the fluid velocity. It is well-known that the role played by the CL becomes particularly important in the theory describing the weakly nonlinear development of perturbations because it is within the critical layer that the nonlinear processes dominate. Since

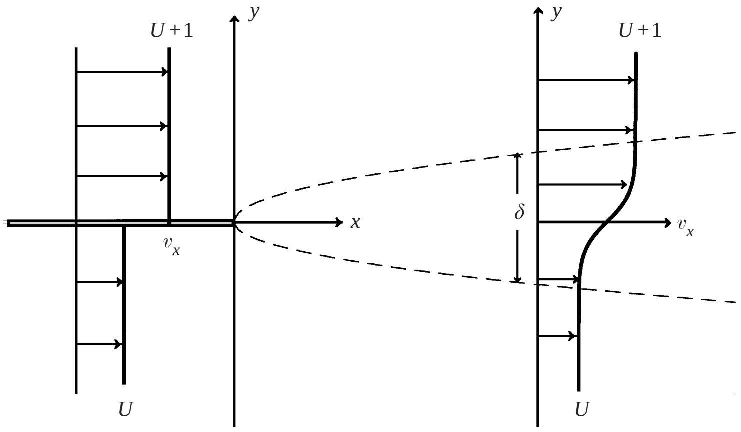


FIGURE 1. Formation of the free mixing layer behind the separating plate under the influence of viscous friction (schematically).

just the viscous effects are responsible for the structure of such CLs (Benney & Bergeron 1969; Haberman 1972; Brown & Stewardson 1978; Churilov & Shukhman 1996) they must of necessity be taken into account (inside the CL at least). On the other hand, taking the viscosity into account leads to the fact that the flow cannot be treated as strictly parallel, and for a consistent construction of a weakly nonlinear theory, one has to take into consideration the weak non-parallelism of the flow. Taking this into account is especially important in problems of the *spatial* evolution of perturbations when their downstream development is investigated (see, for example, Goldstein & Hultgren 1988; Churilov & Shukhman 1994; Shukhman & Churilov 1997). The reason is that viscous spreading effects of the main flow can compete with enhancement effects of perturbations. In principle, if it is proposed to take into account the flow non-parallelism only in the CL region and only over a short length along the flow, it is then possible to restrict ourselves merely to the perturbation method, and to take into account the non-parallelism as a correction to the main parallel flow. This was done in the references just cited. However, a more consistent approach involves constructing a mean flow which includes from the outset the spreading effect, and is constructed based on the initial viscous equations.

Under laboratory conditions, one of the most extensively used methods of producing mixing layers involves merging two parallel streams of nearly inviscid fluid flowing with different velocities on different sides, $y > 0$ and $y < 0$, of a thin separating semi-bounded ($x < 0$) plate (see figure 1). In the region $x > 0$, the flows mix together, and viscous friction gives rise to a transition layer which slightly expands ($\delta \sim (\nu x)^{1/2}$) downstream. Such a mixing layer is sometimes referred to as a free boundary layer.

The solution for an expanding (under the action of viscous forces) mixing layer was originally found by Lock (1935) who applied the Blasius equations (see e.g. Landau & Lifschitz 1987) which were derived for describing a laminar boundary layer on a semi-infinite plate ($x > 0$, $y > 0$), to the problem of a free mixing layer by substituting the boundary condition on the plate (i.e. when $y = 0$) for a boundary condition when $y = -\infty$.

A much more challenging problem involves a consistent construction of the mean flow in the situation where a longitudinal magnetic field comes into play. Models of plane flows with a parallel magnetic field are widely used in magnetohydrodynamic stability theory (see, for example, Drazin & Howard 1966; Kent 1968; Chen & Morrison 1991; Shukhman 1998*a,b*). The reason is that, with the magnetic field present, one further characteristic, the Alfvén velocity, emerges in the problem. Alfvén waves provide an additional carrier of perturbations, and can influence the upstream flow. On this basis, the question of the possibility of existence of stationary solutions with a longitudinal magnetic field becomes non-trivial, and calls for a special investigation. An affirmative answer to this question becomes particularly doubtful in the situation where the Alfvén velocity somewhere exceeds the flow velocity, $c_A > u_{\min}(y)$. However, it is this question that is highly important for a correct statement of the problem of the nonlinear development of disturbances in magnetic mixing layers (for details see the end of § 6).

The non-trivial role of the parameter $\beta = c_A/v$ (characterizing the relationship between electromagnetic forces and inertial forces) was detected in earlier studies on viscous MHD flows of perfectly conducting fluid with a longitudinal magnetic field streaming past bodies (Hasimoto 1959*a*). Specifically it was shown that a super-Alfvén flow ($\beta < 1$) is qualitatively similar to the flow of ordinary fluid, while the sub-Alfvén flow ($\beta > 1$) manifests unusual properties. For instance, a viscous wake which is present in ordinary flows behind the body faces forward in sub-Alfvén flows. Hasimoto (1959*a*) suggested that this effect is associated with Alfvén waves.

The objective of this paper is to generalize Lock's problem of a free mixing layer to the case with a magnetic field, i.e. to study its structure at different values of the magnetic Prandtl number $P_m$ and different parameters of interacting streams, to obtain (if it is possible) explicit analytical expressions for magnetic field and velocity profiles for arbitrary $P_m$, and to ascertain whether stationary sub-Alfvén mixing layers exist.

Various problems concerning the MHD boundary layer at obstacles for different boundary conditions on the surface of a body for different mutual orientations of the magnetic field and velocity, with the shape of the body taken into account, have been considered in a large number of publications. For instance, Sears & Resler (1958) studied the perturbation of a parallel, purely inviscid flow of a nearly perfectly conducting fluid near an infinite sinusoidal wall with a parallel and perpendicular direction of the external magnetic field.

A very similar situation when the magnetic field and the velocity in the external flow are parallel to the surface of a flat plate was studied by Glauter (1962), Gribben (1965, 1967), Ingham (1965), Stewartson (1965), and Meksyn (1962, 1966). As a rule, the solutions are constructed in the form of series in small parameters $P_m = \nu/\nu_m$ or $\beta = c_A/v$. Meksyn (1962, 1967) also studied the structure of the boundary layer at different $P_m$, and found the impossibility of stationary sub-Alfvén boundary layers (the terms Alfvén velocity and flow velocity are used to mean their values in the external flow). The cited references address mainly flows of an incompressible fluid. Allowance for the compressibility complicates the problem considerably, and only a small number of publications exist on this issue (Ingham 1965, 1967).

By considering the appropriately determined magnetic Reynolds number and usual Reynolds number to be large, $Re_m \gg 1$, $Re \gg 1$, and also by limiting attention to the case of an incompressible fluid,† we shall apply, following Lock, the approach used

† Here we confine ourselves only to the simpler case of an incompressible fluid when the sound velocity greatly exceeds both the Alfvén velocity and the flow velocity. Possible effects associated with magnetosonic waves are thereby excluded from consideration. Note that an attempt to generalize Lock's solution to the case of compressible fluid without magnetic field was made by Groppengiesser (1969). However even in the problem without magnetic field he found only one, very specific class of solutions for which the temperature of fluid $T$ is a function only of the $x$-component of velocity,

in studies of the boundary layer, to the problem of the mixing of two free flows with different longitudinal magnetic fields.

It will be shown that super-Alfvén flows exist at any value of $P_m$, and for an arbitrary relation between velocity and magnetic field differences across the layer. On the other hand, sub-Alfvén flows are impossible, at least they cannot be obtained by the method described above. This result for sub-Alfvén flows is rather surprising, especially in the light of the fact that at the two selected values of $P_m$, $P_m = 0$ and $P_m = 1$, sub-Alfvén flows are still possible, hence it would seem that such flows could also exist at least in the range $0 \leqslant P_m \leqslant 1$.

The paper is organized as follows. In § 2 we write the basic equations and recall the known results relevant to the case without a magnetic field. In § 3 we describe the exact solutions which can be obtained in two limiting cases: $P_m = \infty$ (perfect conductivity, frozen magnetic field), and $P_m = 0$ (inviscid fluid). It will be shown that when $P_m = \infty$ only super-Alfvén flows are possible, but when $P_m = 0$, sub-Alfvén flows are also admissible. In § 4 we describe the exact solutions for $P_m = 1$, and demonstrate that sub-Alfvén flows are also possible here. In order to gain an understanding of whether sub-Alfvén flows are possible at an arbitrary $P_m$ (when no exact solutions can be found), in § 5 (and in the Appendix) we undertake an asymptotic analysis for the case of small relative velocity and magnetic field differences. The analysis will show that sub-Alfvén flows are impossible at any $P_m$ (except for the above-mentioned cases $P_m = 0$ and $P_m = 1$), although super-Alfvén flows are possible for any $P_m$ and for any given relations between the magnetic field and velocity differences. These super-Alfvén mixing layers are described by explicit analytic expressions. Some interesting limiting cases are discussed in detail in § 5.1.

In § 6, the results obtained are summarized and an attempt is made to give a physical interpretation of the non-existence of sub-Alfvén mixing layers in the general case and their possibility in the special case $P_m = 1$.

## 2. Basic equations and reference information on the solution with no magnetic field

The system of two-dimensional equations of incompressible magnetic hydrodynamics has the form

$$\{\Delta\psi, \psi\} - \frac{1}{4\pi\rho}\{\Delta A, A\} = \nu\Delta^2\psi, \tag{2.1}$$

$$\{A, \psi\} = \nu_m \Delta A, \tag{2.2}$$

where $\psi(x, y)$ is the stream function, $A(x, y)$ is the $z$-component of the magnetic field vector-potential, $\rho$ is the fluid density, $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$, $\{a, b\} = \partial a/\partial x\, \partial b/\partial y - \partial a/\partial y\, \partial b/\partial x$ and

$$v_x = \frac{\partial\psi}{\partial y}, \quad v_y = -\frac{\partial\psi}{\partial x}, \quad H_x = \frac{\partial A}{\partial y}, \quad H_y = -\frac{\partial A}{\partial x}.$$

Here $v_j$ and $H_j$ are the velocity component and the component of the magnetic field, respectively. In (2.2) it is assumed that electric field is absent, $E_z = 0$.

The transition from the system (2.1), (2.2) to the boundary layer approximation implies that the derivatives with respect to $x$ are considered to be much smaller than

$T = T(v_x)$, while the ordinary Prandtl number $P$ is exactly equal to unity. Nevetheless, this solution has been used to date as a model of background flow in nonlinear stability problems of compressible fluid (Goldstein & Leib 1989; Leib 1991).

the derivatives with respect to $y$, and the velocity and magnetic field $y$-components, respectively, are considered to be smaller than the $x$-components: $v_y/v_x \sim H_y/H_x \sim \delta/l \ll 1$. Here $\delta$ and $l$ are the typical scales of the flow in $y$ and $x$. As a result, instead of (2.1) and (2.2) we have (Vatazhin, Lyubimov & Regirer 1970)

$$\left(\psi' \frac{\partial \psi'}{\partial x} - \psi'' \frac{\partial \psi}{\partial x}\right) - \frac{1}{4\pi\rho}\left(A' \frac{\partial A'}{\partial x} - A'' \frac{\partial A}{\partial x}\right) = \nu \psi''', \tag{2.3}$$

$$\frac{\partial A}{\partial x}\psi' - A' \frac{\partial \psi}{\partial x} = \nu_m A''. \tag{2.4}$$

The prime denotes the derivative with respect to $y$. Note that the transition from (2.1) to (2.3) corresponds to the assumption of a constant total pressure, $p + H^2/8\pi =$ const, in analogy with the usual hydrodynamics where a similar transition signifies a constancy of fluid pressure, $p =$ const.

To simplify the subsequent formulas, we put $(4\pi\rho)^{1/2} = 1$. In these units, the magnetic field has the dimension of velocity.

Our concern is with solutions of the system of equations (2.3) and (2.4) which describe flows such as a mixing layer, that is, flows with boundary conditions

$$v_x = \begin{cases} U, & y \to -\infty \\ U + \Delta U, & y \to +\infty, \end{cases} \qquad H_x = \begin{cases} H_-, & y \to -\infty \\ H_+, & y \to +\infty. \end{cases} \tag{2.5}$$

Further, it is assumed that $U \geqslant 0$, and we choose the value of the velocity difference $\Delta U$ as the velocity unit: $\Delta U = 1$. Since the problem does not involve a typical length parameter $l$, using the same argument as when passing from the Prandtl equation to the Blasius equation (see e.g. Landau & Lifschitz 1987) we conclude that in the region $x > 0$ the solution must be of the form

$$\psi = (\nu x)^{1/2} f(\xi), \quad A = (\nu x)^{1/2} \chi(\xi), \tag{2.6}$$

where $\xi = y/(\nu x)^{1/2}$. In these variables, the velocity and magnetic field components are

$$\left.\begin{aligned} v_x &= f', \quad v_y = \tfrac{1}{2}(\nu/x)^{1/2}(\xi f' - f), \\ H_x &= \chi', \quad H_y = \tfrac{1}{2}(\nu/x)^{1/2}(\xi \chi' - \chi), \end{aligned}\right\} \tag{2.7}$$

where the prime denotes the derivative with respect to $\xi$, and the system (2.3), (2.4) becomes a system of ordinary differential equations:

$$\tfrac{1}{2}(\chi\chi'' - ff'') = f''', \tag{2.8}$$

$$\tfrac{1}{2}(\chi f' - \chi' f) = \frac{1}{P_m}\chi''. \tag{2.9}$$

In the case without a magnetic field, $\chi \equiv 0$, equation (2.8) becomes the well-known Blasius equation. We shall consider $\nu$ and $\nu_m$ to be of the same order, i.e. $P_m \equiv Re_m/Re = O(1)$, although the limiting cases of large and small $P_m$ also are investigated to provide the character of solution behaviour at different $P_m$.

The system (2.8), (2.9) requires five boundary conditions, four of which are formulated using (2.5) and (2.7):

$$f'(-\infty) = U, \quad f'(\infty) = 1 + U, \tag{2.10a}$$

$$\chi'(-\infty) = H_-, \quad \chi'(\infty) = H_+. \tag{2.10b}$$

To determine the fifth boundary condition, note that the system (2.8), (2.9) does

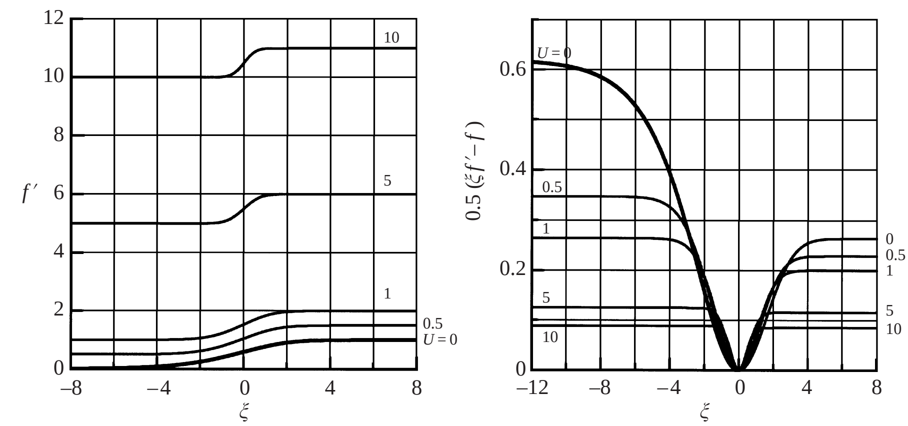


FIGURE 2. The $v_x$- and $v_y$-profiles for the flow with no magnetic field. Numbers labelling the curves indicate a corresponding value of the velocity $U$.

not contain the variable $\xi$ in an explicit form. Therefore, if the pair of functions $f(\xi)$ and $\chi(\xi)$ is the solution of the system with the boundary conditions (2.10), then the pair $f(\xi-\xi_0)$ and $\chi(\xi-\xi_0)$ is also the solution of this system *with the same boundary conditions*. The parameter $\xi_0$ appears to fix the 'centring' of the solution in $\xi$ and it can be chosen only through a detailed matching of the solution from the region of the separating plate ($x<0$) with the solution in the flow region of our interest well away from the plate edge ($x \gg \nu/\Delta U$). We will not concern ourselves with such a matching because of the extreme difficulty of such a task but leave $\xi_0$ as a free parameter, the specification of which provides the missing fifth boundary condition.

It will be recalled that the usual hydrodynamic solution for the mixing layer was obtained by Lock (1935). He integrated the Blasius equation

$$2f''' + ff'' = 0 \tag{2.11}$$

with the particular case of the boundary conditions (2.10*a*), in which $U=0$ (i.e. $f'(-\infty)=0$, $f'(\infty)=1$). As the third boundary condition, he used the vanishing of the velocity $y$-component when $\xi=0$: $v_y(0) \propto (f'\xi - f)_{\xi=0} = 0$, i.e. $f(0)=0$. Since Lock's paper is not readily available (although presented in detail in the somewhat more easily available report of Gropengiesser 1969), we reproduced its results by supplementing them by our own numerical calculations for $U \neq 0$.

Figure 2 shows the profiles of the functions $f'(\xi)$ and $0.5(\xi f' - f)$, with which the velocity components $v_x$ and $v_y$ are connected (see (2.7)), at different $U$. The curve with $U=0$ corresponds to the solution obtained by Lock (1935).

In the case $U \gg 1$ one can obtain the analytical solution

$$\left.\begin{aligned} f &= \{U + \tfrac{1}{2}[1+\operatorname{erf}(\tfrac{1}{2}\xi\sqrt{U})]\}\xi - (\pi U)^{-1/2}[1-\exp(-\tfrac{1}{4}U\xi^2)], \\ v_x &= U + \tfrac{1}{2}[1+\operatorname{erf}(\tfrac{1}{2}\xi\sqrt{U})], \end{aligned}\right\} \tag{2.12}$$

where $\operatorname{erf}(x) = (2/\sqrt{\pi})\int_0^x \exp(-t^2)\,\mathrm{d}t$ is the probability integral.

The flow (2.12) is called *Blasius mixing layer*, and often it is used as a standard model in stability investigations (even for flows with $\Delta U/U$ not small).

The solutions of (2.11) when $U = O(\Delta U) = O(1)$ describe an expanding mixing layer, in which $\delta/x \sim v_y/v_x \sim (\nu/x)^{1/2} \approx Re^{-1/2} \ll 1$, if the local Reynolds number is defined as $Re = x\Delta U/\nu \equiv x/\nu \gg 1$. (For solutions with a relatively small velocity difference $\Delta U/U \equiv U^{-1} \ll 1$ we have $\delta/x \sim (\nu/xU)^{1/2}$, i.e. again $\delta/x \sim Re^{-1/2}$, because it is more reasonable here to define the Reynolds number as $Re(x) = xU/\nu$.)

## 3. Flows with a magnetic field: the cases of perfect ($P_m \to \infty$) conductivity and ideal ($P_m = 0$) fluid

In the general case the system of equations (2.8) and (2.9) is rather complicated and must be solved numerically. In three special cases, however, it is relatively easy to analyse its solutions analytically. First, is the case where the magnetic viscosity is much smaller than the usual viscosity, $P_m \to \infty$, arbitrarily referred to as the perfect conductivity case. Second, is the case when the usual viscosity is much smaller than the magnetic viscosity, $P_m \to 0$, referred to as the inviscid fluid case. Third, is an intermediate case where the magnetic viscosity is exactly equal to the usual viscosity, $P_m = 1$. In this Section we shall consider the first two cases, and the third case will be detailed in the next Section.

### 3.1. *The case* $P_m \to \infty$

The right-hand side of (2.9) becomes zero, and we have $\chi' f - \chi f' = 0$. From this and the boundary conditions (2.10) it follows that

$$\chi(\xi) = \beta f(\xi), \quad \beta = \text{const} = \frac{H_-}{U} = \frac{H_+}{1+U}. \tag{3.1}$$

It is evident that solution is not possible with an arbitrary relationship of values of the magnetic field on different sides of the layer but only when the condition

$$\Delta H \equiv H_+ - H_- = \beta \tag{3.2}$$

is satisfied. Also, the ratio of the Alfvén velocity to the flow velocity is the same at all points of the stream and equals $\beta$: $c_A(\xi)/v_x(\xi) \equiv H_x(\xi)/v_x(\xi) = \beta$. On substituting (3.1) into (2.8), we obtain

$$2f''' + (1-\beta^2) f f'' = 0. \tag{3.3}$$

Substituting $\tilde{\xi} = \mu^{1/2}\xi$, $\tilde{f} = \mu^{1/2} f$, where $\mu \equiv 1 - \beta^2$, reduces equation (3.3) to the Blasius equation (2.11) with the boundary conditions (2.10*a*). Therefore, $\tilde{f}(\tilde{\xi}) = f_L(\tilde{\xi})$ and

$$f(\xi) = \mu^{-1/2} f_L(\mu^{1/2}\xi), \tag{3.4}$$

where $f_L(\xi)$ is any solution of equation (2.11) with the boundary conditions (2.10*a*). From (3.4) it follows that the layer thickness

$$\delta \sim (\nu x/\mu)^{1/2} \tag{3.5}$$

in the presence of a magnetic field, $\beta \neq 0$, is larger than in the case with no magnetic field, and tends to infinity when the Alfvén velocity tends to the flow velocity, i.e. when $\beta \to 1$. With an increase in layer thickness, we are outside the validity range of the basic equations describing quasi-parallel flows with quasi-parallel magnetic fields. One can also check that when $\beta > 1$, equation (3.3) does not have any solutions at all with the required boundary conditions. This means that sub-Alfvén ($v_x < c_A$) flows of such a type with a frozen-in magnetic field are impossible.

Note that an unambiguous connection between the magnetic field difference and the velocity difference (3.2) arises only within the perfect conductivity approximation and when the requirement of continuity of magnetic field is imposed. In principle, otherwise infinitely thin current sheets are possible. For example such surface currents appear at the walls of streamlined obstacles in perfectly conducting fluid (Sears & Resler 1958). It will be shown in § 5 that in the mixing layer a thin current sheet also appears if $\Delta H \neq \beta$, and its width is defined by large but finite conductivity, $\Delta y \propto (\nu_m)^{1/2}$. In other words, inclusion of an arbitrary large but finite value of $P_m$ (an arbitrary small magnetic viscosity) makes possible a continuous super-Alfvén solution with an arbitrary $\Delta H$ and not only with $\Delta H = \beta$. In this case the magnetic field profiles show a narrow 'jump' of size $\Delta H - \beta$. Nevertheless, it will be shown that even the abandonment of freezing-in does not lead to the possibility of sub-Alfvén flows.

### 3.2. *The case* $P_m = 0$

In this case it is more fruitful to introduce instead of (2.6) self-similar variables based on magnetic viscosity $\nu_m$:

$$\psi = (\nu_m x)^{1/2} f(\xi), \quad A = (\nu_m x)^{1/2} \chi(\xi), \quad \xi = y/(\nu_m x)^{1/2}. \tag{3.6}$$

Now we have

$$\left.\begin{aligned} v_x &= f', \quad v_y = \tfrac{1}{2}(\nu_m/x)^{1/2}(\xi f' - f), \\ H_x &= \chi', \quad H_y = \tfrac{1}{2}(\nu_m/x)^{1/2}(\xi\chi' - \chi), \end{aligned}\right\} \tag{3.7}$$

and instead of (2.8) and (2.9) we obtain

$$\tfrac{1}{2}(\chi\chi'' - ff'') = P_m f''' = 0, \tag{3.8}$$

$$\tfrac{1}{2}(\chi f' - \chi' f) = \chi''. \tag{3.9}$$

This system has an obvious partial solution which describes the well-known purely parallel vortex sheet (with unity velocity jump) in a uniform magnetic field:

$$\chi = H_0\xi, \quad H_x = H_0, \quad H_y = 0, \tag{3.10}$$

$$f = U\xi + \tfrac{1}{2}(\xi + |\xi|), \quad v_x = U + \tfrac{1}{2}(1 + \operatorname{sign}(\xi)), \quad v_y = 0. \tag{3.11}$$

In this solution the Alfvén velocity $c_A(= H_0)$ can be both smaller and larger than the flow velocity.

This conclusion is fairly obvious because inviscid pure longitudinal flow in the uniform magnetic field does not feel the ponderomotive force, i.e. there is no interaction between the conducting fluid and magnetic field. Hence the ratio between Alfvén and flow velocities in this very specific case may be arbitrary.

However, we shall see in § 5 that the statement about the possibility of sub-Alfvén flow for inviscid fluid is valid only for the case of strongly uniform field, $\Delta H = 0$. If $\Delta H \neq 0$ the diffusive spreading of the magnetic profile generates an $H_y$-component and a ponderomotive force appears. As a result inviscid stationary sub-Alfvén flows become prohibited. We come to the same conclusion if arbitrarily small viscosity is permitted – then the appearance of $v_y$ due to viscous flow spreading also induces the force.

## 4. Exact solutions for the case of equal viscosities ($P_m = 1$)

It is convenient to deal, instead of (2.9), with equations obtained as a result of its differentiation. Then from (2.8) and (2.9) we obtain the system

$$f''' = \tfrac{1}{2}(\chi\chi'' - ff''), \quad \chi''' = \tfrac{1}{2}(\chi f'' - f\chi''). \tag{4.1}$$

Also, it is useful to bear in mind that the second equation in (4.1) has the integral

$$\chi'' - \tfrac{1}{2}(\chi f' - \chi' f) = \text{const} = 0. \tag{4.2}$$

It is convenient to introduce the functions $g_1 = f - \chi$, $g_2 = f + \chi$, so that

$$f = \tfrac{1}{2}(g_2 + g_1), \quad \chi = \tfrac{1}{2}(g_2 - g_1). \tag{4.3}$$

From (4.1) we obtain

$$g_1''' = -\tfrac{1}{2} g_2 g_1'', \quad g_2''' = -\tfrac{1}{2} g_1 g_2''. \tag{4.4}$$

Boundary conditions for (4.4) follow from (2.10) and (4.2):

$$\left.\begin{array}{l} g_1'(-\infty) = U - H_-, \ \ g_1'(+\infty) = (1+U) - H_+, \\ g_2'(-\infty) = U + H_-, \ \ g_2'(+\infty) = (1+U) + H_+, \\ (g_1' g_2 - g_2' g_1)_{\pm\infty} = 0. \end{array}\right\} \tag{4.5}$$

Note that the system (4.4) is invariant with respect to the substitution $g_1 \to g_2$, $g_2 \to g_1$. This is a reflection of the obvious property of the initial system (2.8) and (2.9) implying that if the pair $f$ and $\chi$ is a solution of this system, then the pair $f$ and $-\chi$ is also a solution of this system (with the appropriately modified boundary conditions (2.10*b*) for $\chi$: $\chi(\pm\infty) = -H_\pm$). In other words, a change of sign of the magnetic field does not influence the flow.

The system (4.4) allows two simple partial solutions. The first can be obtained if we put $g_2'' \equiv 0$, and the second solution is obtainable by putting $g_1'' \equiv 0$. We obtain an explicit form of solution in these two cases.

(i)

$$g_2'' = 0, \quad g_2' = \text{const} = 4\lambda_1, \quad g_2 = 4\lambda_1(\xi - \xi_0). \tag{4.6}$$

In (4.6), $\xi_0$ is an arbitrary constant (see comments in §2), and the constant $\lambda_1$ is determined from the boundary conditions (4.5):

$$\lambda_1 = \tfrac{1}{4}(U + H_-) = \tfrac{1}{4}(1 + U + H_+). \tag{4.7}$$

The relation (4.7) shows that in this solution the sign of the magnetic field difference is opposite to the sign of the velocity difference and equal to it in absolute value:

$$H_+ = H_- - 1 \quad \text{or} \quad \Delta H = -1. \tag{4.8}$$

Substitute (4.6) into the first equation (4.4). Designating $\xi - \xi_0 = \zeta$, we obtain

$$g_1''' = -2\lambda_1 \zeta g_1''. \tag{4.9}$$

Integrating (4.9) gives

$$f = (U + \tfrac{1}{2})\zeta + (\lambda_1/\pi)^{1/2} \int_0^{\zeta} \mathrm{d}z_1 \int_0^{z_1} \mathrm{d}z_2\, \mathrm{e}^{-\lambda_1 z_2^2} + \tfrac{1}{2}(\pi\lambda_1)^{-1/2}, \tag{4.10$a$}$$

$$\chi = -f + (U + H_-)\zeta. \tag{4.10$b$}$$

(ii) $g_1'' = 0$, $g_2' = \text{const} = 4\lambda_2$, $g_1 = 4\lambda_2(\xi - \xi_0)$, and

$$\lambda_2 = \tfrac{1}{4}(U - H_-) = \tfrac{1}{4}(1 + U - H_+). \tag{4.11}$$

Here the magnetic field difference is equal to the velocity difference:

$$H_+ = H_- + 1 \quad \text{or} \quad \Delta H = 1. \tag{4.12}$$

On performing the same procedure as above, we find

$$f = (U + \tfrac{1}{2})\,\zeta + (\lambda_2/\pi)^{1/2}\int_0^{\zeta} \mathrm{d}z_1 \int_0^{z_1} \mathrm{d}z_2\, \mathrm{e}^{-\lambda_2 z_2^2} + \tfrac{1}{2}(\pi\lambda_2)^{-1/2}, \tag{4.13a}$$

$$\chi = f - (U - H_-)\zeta. \tag{4.13b}$$

We can calculate the layer thickness for these solutions, which is defined by the relation $\delta = \int_{-\infty}^{\infty} \mathrm{d}y\,(v_x - U)(1 + U - v_x)$:

$$\delta_{1,2}(x) = (\nu x/2\pi\lambda_{1,2})^{1/2}, \quad \lambda_{1,2} = \tfrac{1}{4}(U \pm H_-). \tag{4.14}$$

Let us call the solution with $\lambda_1$ (i.e. (4.10)) the 'narrow' solution and the solution with $\lambda_2$ (i.e. (4.13)) the 'wide' solution, because when $H_- > 0$, in accordance with (4.14), $\delta_1 < \delta_2$.

From the conditions $\lambda_{1,2} > 0$ it follows that the narrow solution is possible only when $H_- > -U$ and the wide solution is possible when $H_- < U$. Since, as has already been pointed out above, the solutions $(f, \chi)$ and $(f, -\chi)$ represent the same solution from the dynamic standpoint, it will suffice to confine ourselves only to non-negative values of $H_-$, $H_- \geqslant 0$.

It is easy to see that the wide solution is super-Alfvén at each point of the flow, while variants are possible for the narrow solution. When $H_- > U + 2$, the flow in the entire stream is sub-Alfvén, and when $H_- < U$ it is super-Alfvén. When $U < H_- < U + 2$, the profile involves both super-Alfvén ($\xi > \xi_c$) and sub-Alfvén ($\xi < \xi_c$) regions.

Note that when $0 < H_- < 1$ the $x$-component of the magnetic field changes its sign. On the other hand, since the solution with a change of sign violates the applicability conditions of the initial system (2.8), (2.9) (the quasi-parallelism is violated in the neighbourhood of the magnetic field zero line) the narrow solution with $0 \leqslant H_- \leqslant 1$ must be excluded from the set of possible solutions.

Let us consider the behaviour of the velocity and magnetic field components. Using (2.7), (4.10) and (4.13) we write

$$v_x = U + \tfrac{1}{2}[1 + \operatorname{erf}(t - t_0)], \quad v_y = \tfrac{1}{2}(\nu/x\lambda_{1,2})^{1/2} V_y(U, t_0; t), \tag{4.15}$$

$$H_x = H_- \mp \tfrac{1}{2}[1 + \operatorname{erf}(t - t_0)], \quad H_y = \tfrac{1}{2}(\nu/x\lambda_{1,2})^{1/2} h_y(H_-, t_0; t). \tag{4.16}$$

Here $t = \lambda_{1,2}\xi$, $t_0 = \lambda_{1,2}\xi_0$, and

$$V_y(U, t_0; t) = (U + \tfrac{1}{2})\, t_0 + \tfrac{1}{2} t_0 \operatorname{erf}(t - t_0) - \tfrac{1}{2}\pi^{-1/2}\mathrm{e}^{-(t-t_0)^2}, \tag{4.17}$$

$$h_y(H_-, t_0; t) = H_- t_0 \mp \tfrac{1}{2}[t_0 + t_0 \operatorname{erf}(t - t_0) - \pi^{-1/2}\mathrm{e}^{-(t-t_0)^2}]. \tag{4.18}$$

In (4.15)–(4.18), the upper sign and the index 1 refer to the narrow solution, and the lower sign and the index 2 refer to the wide solution.

The velocity and magnetic field profiles calculated numerically for some sets of parameters are presented in figures 3–5: Figure 3 shows the profiles of the functions $v_x(U, t_0; t)$ and $V_y(U, t_0; t)$ for two values of $U$, $U = 0$ and $U = 1$, and for several values of the parameter $t_0$. It will be recalled that the parameter $t_0 (\equiv \lambda_{1,2}\xi_0)$ determines the centring of the solution. Profiles of the magnetic field components are shown in figures 4 and 5. Note, that only if $\xi_0 = 0$ do $y$-components of magnetic field and velocity vanish on both sides of the layer.

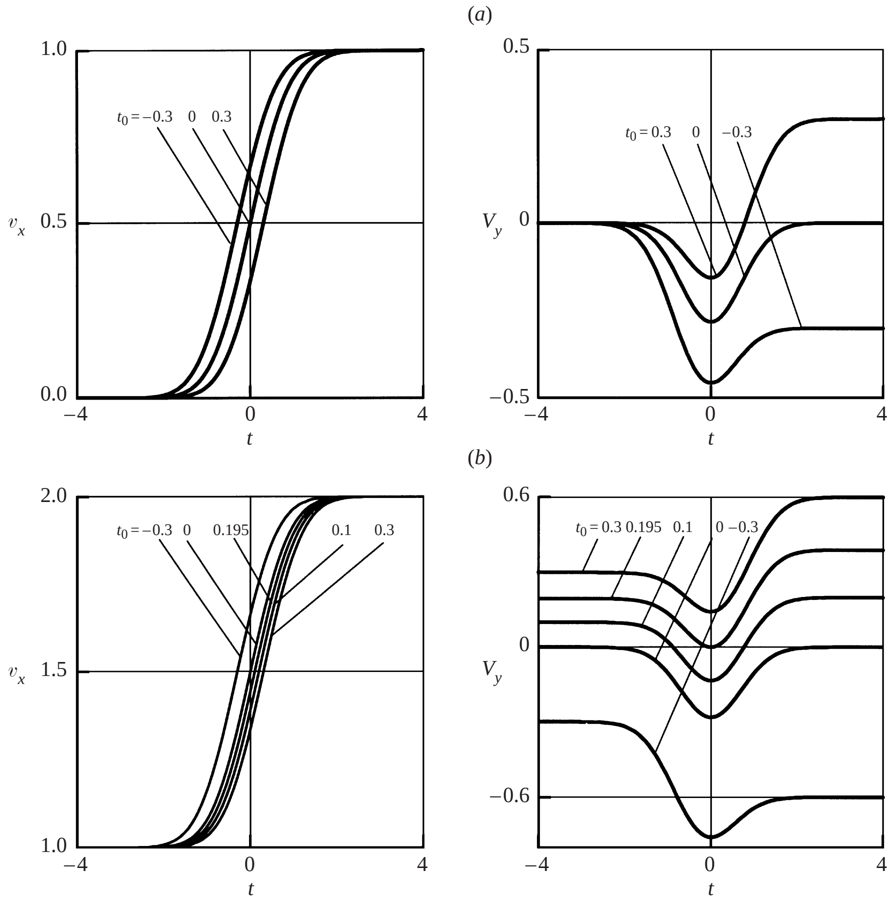


FIGURE 3. Profiles of the $x$-component of the velocity $v_x$ and the reduced $y$-component of the velocity $V_y = 2v_y(x\lambda_{1,2}/\nu)^{1/2}$ for several values of the 'centring' parameter $t_0$ and for two values of the velocity $U$ for solutions with $P_m = 1$: ($a$) $U = 0$, $t_0 = -0.3, 0, 0.3$; ($b$) $U = 1$, $t_0 = -0.3, 0, 0.1, 0.195, 0.3$. The horizontal axis indicates the reduced self-similar variable $t = \xi/(\lambda_{1,2})^{1/2}$. (The value of $H_-$ here is arbitrary and does not influence the form of the velocity profiles in reduced variables.)

## 5. The case of arbitrary $P_m$: flows with small relative velocity and magnetic field differences

As has been shown in the preceding sections, sub-Alfvén flows are possible when $P_m = 0$ and $P_m = 1$. It might be expected at first glance that they could also exist at intermediate values of $P_m$ at least. In order to check whether this is the case, we now examine the case of arbitrary $P_m$ in greater detail. Since the system (2.8), (2.9) at arbitrary $P_m$ defies analytical investigation in a general form, we consider the situation when the problem involves a small parameter. We shall assume the relative velocity and magnetic field difference to be such a parameter. Then at the main order of perturbation theory the system (2.8), (2.9) becomes linear in this parameter, and we can construct a solution in the form of power series of this small parameter.

Thus, let $\Delta U/U \equiv 1/U = \varepsilon \ll 1$, $\Delta H/H_- = O(\varepsilon)$ and assume that the magnetic field and the velocity are quantities of the same order: $H_- = \beta U$, where $\beta = O(1)$, and, without loss of generality, we assume that $\beta > 0$. In this notation, values of $\beta < 1$ correspond to super-Alfvén flows, and values of $\beta > 1$ correspond to sub-Alfvén

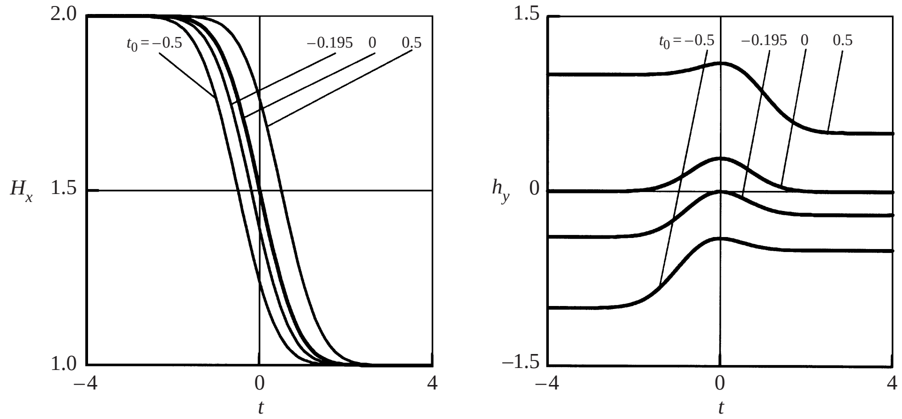


FIGURE 4. 'Narrow' solution. Profiles of the $x$-component of the magnetic field $H_x$ and the reduced $y$-component of field $h_y = 2H_y(x\lambda_1/\nu)^{1/2}$ when $H_- = 2$ for four values of the parameter $t_0$ ($t_0 = -0.5, -0.195, 0, 0.5$). The horizontal axis indicates the variable $t = \xi/(\lambda_1)^{1/2}$. (The value of the velocity $U$ here can be arbitrary and does not influence the form of field profiles in reduced variables.)

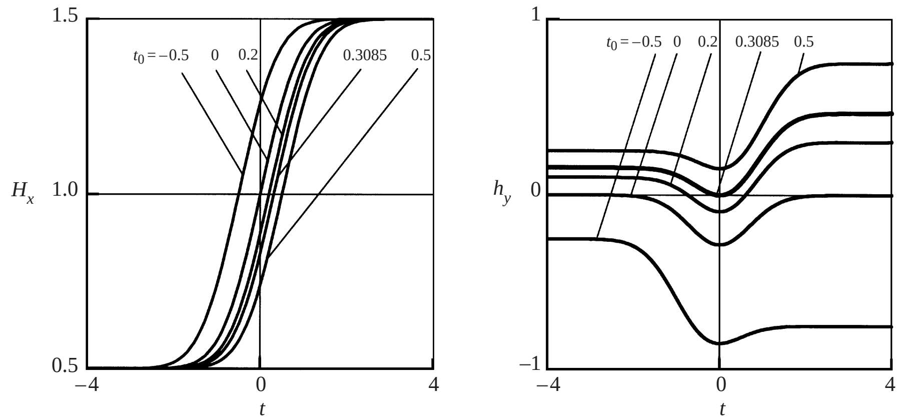


FIGURE 5. 'Wide' solution. Profiles of the $x$-component of the magnetic field $H_x$ and of the reduced $y$-component of field $h_y = 2H_y(x\lambda_2/\nu)^{1/2}$ when $H_- = 0.5$ and $U > 0.5$ for five values of the parameter $t_0$ ($t_0 = -0.5, 0, 0.2, 0.3085, 0.5$). The horizontal axis indicates the variable $t = \xi/(\lambda_2)^{1/2}$.

flows. The asymptotic theory developed below holds if $\beta$ is not too close to unity: $|1-\beta|\,U \gg 1$.

It is clear that with this limitation the flow is everywhere (i.e. at all $\xi$) either sub-Alfvén or super-Alfvén, because this inequality implies that $|U - H_-| \gg \Delta H \sim \Delta U = 1$, and therefore the quantity $H_x - v_x (= c_A - v_x)$ cannot change sign at any point of the profile.

We put

$$f = U\zeta + F, \quad \chi = (U\beta)\,\zeta + G, \tag{5.1}$$

where $\zeta = \xi - \xi_0$, $\xi_0$ being an arbitrary parameter fixed by the fifth boundary condition (see § 2). On substituting (5.1) into (2.8) and into the equation obtained by differentiating (2.9), we obtain

$$\left.\begin{aligned}\tfrac{1}{2}[U\zeta(\beta G'' - F'') + GG'' - FF''] &= F''',\\ \tfrac{1}{2}[U\zeta(\beta F'' - G'') + GF'' - FG''] &= \frac{1}{P_m}G'''.\end{aligned}\right\} \tag{5.2}$$

It is assumed that the prime in (5.2) denotes the derivative with respect to $\zeta$. Six boundary conditions for the system (5.2) are

$$F' = \begin{cases} 0, \zeta \to -\infty \\ 1, \zeta \to +\infty, \end{cases} \qquad G' = \begin{cases} 0, & \zeta \to -\infty \\ \Delta H, & \zeta \to +\infty, \end{cases} \qquad F = G = 0 \quad \text{when} \quad \zeta \to -\infty. \tag{5.3}$$

Since $U = \varepsilon^{-1}$ is a large parameter, the system (5.2) should be solved by the method of matched asymptotic expansions. The outer solution occurs in the regions $\varepsilon^{1/2} \lesssim |\zeta| < \infty$, and the inner solution holds in the region $|\zeta| \lesssim \varepsilon^{1/2}$. The problem should be solved separately in the outer regions and in the inner region, and should be matched in the region $|\zeta| \sim \varepsilon^{1/2}$ where the solutions overlap.

In the *outer* regions (with the proviso that $|1-\beta| \gg 1/U$) we have

$$\beta G'' - F'' = 0, \quad \beta F'' - G'' = 0,$$

whence, in view of the boundary conditions (5.3), it follows that

$$F = \begin{cases} 0, & \zeta < 0 \\ \zeta + \varepsilon^{1/2}\alpha, & \zeta > 0, \end{cases} \qquad G = \begin{cases} 0, & \zeta < 0 \\ \Delta H\zeta + \varepsilon^{1/2}\gamma, & \zeta > 0, \end{cases} \tag{5.4}$$

where $\alpha$ and $\gamma$ are constants which must be determined from the matching with the inner problem.

We now turn to the *inner* problem and put $\zeta = \varepsilon^{1/2}Z$, $F = \varepsilon^{1/2}\tilde{F}$, $G = \varepsilon^{1/2}\tilde{G}$.

From (5.2) we obtain

$$\left.\begin{aligned}\tilde{F}''' - \tfrac{1}{2}Z(\beta\tilde{G}'' - \tilde{F}'') &= \tfrac{1}{2}\varepsilon(\tilde{G}\tilde{G}'' - \tilde{F}\tilde{F}''),\\ \frac{1}{P_m}\tilde{G}''' - \tfrac{1}{2}Z(\beta\tilde{F}'' - \tilde{G}'') &= \tfrac{1}{2}\varepsilon(\tilde{G}\tilde{F}'' - \tilde{F}\tilde{G}''),\end{aligned}\right\} \tag{5.5}$$

where the prime now denotes the derivative with respect to $Z$. The solution of the system (5.5) must be matched with inner asymptotic representations of the outer solution. From (5.4) we obtain

$$\tilde{F} = \begin{cases} 0, & Z \to -\infty \\ Z + \alpha, & Z \to +\infty, \end{cases} \qquad \tilde{G} = \begin{cases} 0, & Z \to -\infty \\ \Delta H\, Z + \gamma, & Z \to +\infty. \end{cases} \tag{5.6}$$

Following perturbation theory, we seek a solution in the form

$$\tilde{F} = \tilde{F}_0 + \varepsilon\tilde{F}_1 + \ldots, \quad \tilde{G} = \tilde{G}_0 + \varepsilon\tilde{G}_1 + \ldots, \quad \alpha = \alpha_0 + \varepsilon\alpha_1 + \ldots,$$
$$\gamma = \gamma_0 + \varepsilon\gamma_1 + \ldots .$$

By denoting $\tilde{F}_0'' = \Phi_0$, $\tilde{G}_0'' = \Gamma_0$, from (5.5) at the zeroth order in $\varepsilon$ we have

$$\Phi_0' - \tfrac{1}{2}Z(\beta\Gamma_0 - \Phi_0) = 0, \quad \frac{1}{P_m}\Gamma_0' - \tfrac{1}{2}Z(\beta\Phi_0 - \Gamma_0) = 0. \tag{5.7}$$

The system (5.7) can readily be reduced to a single equation. By expressing $\Gamma_0$ from the first equation of (5.7), $\Gamma_0 = \beta^{-1}(2\Phi_0'/Z + \Phi_0)$, and substituting it into the second equation, we obtain

$$\mathscr{L}\Phi_0 = 0, \quad \mathscr{L} = W\frac{\mathrm{d}}{\mathrm{d}Z}\left(\frac{1}{W}\frac{\mathrm{d}}{\mathrm{d}Z}\right) + \tfrac{1}{4}P_m(1-\beta^2)Z^2. \tag{5.8}$$

Here $W(Z) = 2Z(\lambda_2 - \lambda_1)\exp[-(\lambda_2 + \lambda_1)Z^2]$ is the Wronskian of two linearly independent solutions of equation (5.8), $\Phi_N(Z)$ and $\Phi_W(Z)$, where

$$\Phi_N(Z) = \exp(-\lambda_1 Z^2), \quad \Phi_W(Z) = \exp(-\lambda_2 Z^2), \tag{5.9}$$

and

$$\lambda_{1,2} = \tfrac{1}{8}(1 + P_m)\left\{1 \pm \left[1 + \frac{4P_m}{(1 + P_m)^2}(\beta^2 - 1)\right]^{1/2}\right\} \tag{5.10}$$

are the roots of the characteristic equation $(1 - 4\lambda)(1 - 4\lambda/P_m) = \beta^2$. Since $\lambda_1 > \lambda_2$, the two linearly independent solutions $\Phi_N$ and $\Phi_W$ will be called, as done in § 4, the narrow and wide solutions, respectively. It is important to note that $\lambda_1 > 0$ at any $\beta$, and $\lambda_2 > 0$ only for super-Alfvén flows, $\beta < 1$. When $\beta > 1$ we have $\lambda_2 < 0$, and the solution $\Phi_W$ should be discarded.

Consider the cases $\beta < 1$ and $\beta > 1$ separately.

### 5.1. *Super-Alfvén flows,* $\beta < 1$

When $\beta < 1$ we seek the solution of (5.8) in the form

$$\Phi_0(Z) = A_N\Phi_N(Z) + A_W\Phi_W(Z). \tag{5.11}$$

In this case

$$\Gamma_0 = \beta^{-1}[(1 - 4\lambda_1)A_N\Phi_N + (1 - 4\lambda_2)A_W\Phi_W]. \tag{5.12}$$

Upon integrating (5.11) and (5.12) and matching to inner asymptotic representations of the outer solution (5.8) we obtain the expressions for the constant $A_N$ and $A_W$

$$A_N = (\lambda_1/\pi)^{1/2} a_N, \quad A_W = (\lambda_2/\pi)^{1/2} a_W, \tag{5.13}$$

where

$$a_N = \frac{(1 - \beta\Delta H)/4 - \lambda_2}{\lambda_1 - \lambda_2}, \quad a_W = \frac{\lambda_1 - (1 - \beta\Delta H)/4}{\lambda_1 - \lambda_2}, \tag{5.14}$$

as well as the values of the constant $\alpha_0$ and $\gamma_0$ involved in the outer solution: $\alpha_0 = \gamma_0 = 0$.

We now can write the final expressions for the velocity $v_x$ and the magnetic field $H_x$ in original variables:

$$\left.\begin{aligned} v_x &= U + \tfrac{1}{2}\{1 + a_N \operatorname{erf}(\zeta\sqrt{\lambda_1 U}) + a_W \operatorname{erf}(\zeta\sqrt{\lambda_2 U})\}, \\ H_x &= H_- + \tfrac{1}{2}\left\{\Delta H + \frac{1 - 4\lambda_1}{\beta} a_N \operatorname{erf}(\zeta\sqrt{\lambda_1 U}) + \frac{1 - 4\lambda_2}{\beta} a_W \operatorname{erf}(\zeta\sqrt{\lambda_2 U})\right\}. \end{aligned}\right\} \tag{5.15}$$

Hence it is evident that for the case of a super-Alfvén flow, it is possible to find the solution satisfying all boundary conditions imposed. The analytical solution obtained may be considered as a direct generalization of the solution (2.12) for a Blasius mixing layer.

Let us discuss the properties of this solution. As would be expected, generally the flow is described by two scales, $\delta_N \sim (\nu x/\lambda_1 U)^{1/2}$ and $\delta_W \sim (\nu x/\lambda_2 U)^{1/2}$, and generally speaking both the velocity profile and the magnetic field profile include both characteristic scales.

When the Alfvén velocity approaches the flow velocity, i.e. when $1 - \beta \ll 1$, we have

$$\lambda_1 \approx \tfrac{1}{4}(1 + P_m), \quad \lambda_2 \approx \tfrac{1}{4}\mu P_m/(1 + P_m), \quad \text{where} \quad \mu \equiv 1 - \beta^2.$$

The narrow scale $\delta_N \sim \{[\nu_m \nu/(\nu_m + \nu)]\, x/U\}^{1/2}$ in this case becomes substantially smaller than the wide scale $\delta_W \sim [(\nu_m + \nu)x/\mu U]^{1/2}$. The streamlines and magnetic field lines become increasingly curved, the flow becomes progressively less quasi-parallel, and when $1 - \beta \sim 1/U$ the approximation is no longer applicable. What happens to the solution when $\beta > 1$ will be discussed in the next subsection.

In limiting cases of a small magnetic field, $\beta \ll 1$, and also in cases of widely differing viscosities, $P_m \gg 1$ and $P_m \ll 1$, each of the scales is determined by a single viscosity only. Respectively, the magnetic field profile or the velocity profile can be determined by only one scale. We consider these limiting cases as well as the case $P_m = 1$ in greater detail. (This is also instructive in the context of a comparison of the resulting solutions with those obtained in §§ 2–4.)

5.1.1. *Weak magnetic field, $\beta \ll 1$*

From (5.10) we have $\lambda_1 \approx \frac{1}{4}$, $\lambda_2 \approx \frac{1}{4}P_m$ if $P_m < 1$, and $\lambda_1 \approx \frac{1}{4}P_m$, $\lambda_2 \approx \frac{1}{4}$ if $P_m > 1$. It is evident that the larger scale here is determined by a larger viscosity, and the velocity profile and the magnetic field profile are related only to the respective viscosity (i.e. to the usual and magnetic viscosities, respectively). Indeed, from (5.15) we have

$$\left.\begin{aligned} f &= \{U + \tfrac{1}{2}[1 + \operatorname{erf}(\tfrac{1}{2}\zeta\sqrt{U})]\}\zeta + (\pi U)^{-1/2}\exp(-\tfrac{1}{4}U\zeta^2), \\ v_x &= U + \tfrac{1}{2}[1 + \operatorname{erf}(\tfrac{1}{2}\zeta\sqrt{U})], \quad H_x = H_- + \tfrac{1}{2}\Delta H[1 + \operatorname{erf}(\tfrac{1}{2}\zeta\sqrt{P_m U})]. \end{aligned}\right\} \tag{5.16}$$

In this limit the magnetic field has virtually no effect on the flow; therefore, the velocity field here is the same as in the case without a magnetic field. Indeed, it is easy to see that the solution (5.16) for the stream function $f$ and the velocity $v_x$ coincides with the solution (2.12) for the case without a magnetic field if in (5.16) the parameter $\xi_0$ is defined by the value of $\xi_0 = (\pi U^3)^{-1/2}$. (This corresponds to selecting the fifth boundary condition in the form $f(0) = 0$ which was used in § 2.)

5.1.2. *Small magnetic viscosity, $P_m \gg 1$: current sheet*

Here $\lambda_1 \approx \frac{1}{4}P_m$, $\lambda_2 \approx \frac{1}{4}\mu$, and from (5.15) we obtain

$$\left.\begin{aligned} v_x &= U + \tfrac{1}{2}[1 + \operatorname{erf}(\tfrac{1}{2}\zeta\sqrt{U\mu})] + O(P_m^{-1}), \\ H_x &= H_- + \tfrac{1}{2}[\Delta H + (\Delta H - \beta)\operatorname{erf}(\tfrac{1}{2}\zeta\sqrt{P_m U}) + \beta\operatorname{erf}(\tfrac{1}{2}\zeta\sqrt{U\mu})] \\ &\approx H_- + \tfrac{1}{2}[\Delta H + (\Delta H - \beta)\operatorname{sign}(\zeta) + \beta\operatorname{erf}(\tfrac{1}{2}\zeta\sqrt{U\mu})]. \end{aligned}\right\} \tag{5.17}$$

The scale $\delta_N \sim (\nu_m x/U)^{1/2}$ in this case is determined by magnetic viscosity, and it is very small compared with the scale $\delta_W \sim (\nu x/U)^{1/2}$ which is determined by the usual viscosity only. The velocity profile is determined in fact only by the wide (i.e. based on the usual viscous) scale $\delta_W$, while the magnetic field profile is two-scaled. The magnetic field profile at $\zeta \approx 0$ includes a very abrupt (of width $\Delta\zeta \sim (P_m U)^{-1/2}$) jump (a current sheet): $H(+0) - H(-0) = \Delta H - \beta$. Notice that the velocity profile is monotonic only when $\Delta H - \beta \geqslant 0$. When $\Delta H = \beta$, the contribution of the narrow solution, and also the jump in the magnetic field profile disappear, and the magnetic field profile, like the velocity profile, becomes smooth. Furthermore, the velocities and the magnetic field at each point are related by the relation $H_x = \beta v_x$, in full accord with the solution obtained in § 3.1 (see (3.2)).

Thus, comparing these results with those described in § 3.1 we can say that if $\Delta H \neq \beta$ the flow with almost frozen magnetic field produces thin current sheet, where the frozen-in condition is not valid, and just the existence of this sheet provides the required magnetic field difference.

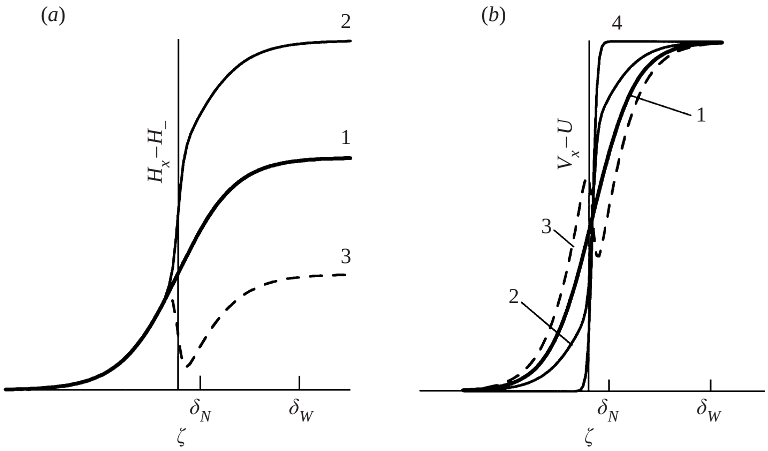


FIGURE 6. Schematical profiles of ($a$) $H_x$ for the case of small magnetic viscosity, $P_m \gg 1$ and ($b$) $v_x$ for the case of small ordinary viscosity, $P_m \ll 1$. ($a$) Curve 1: $\Delta H = \beta$, curve 2: $\Delta H > \beta$ and curve 3: $\Delta H < \beta$ (non-monotonic profile); ($b$) curve 1: $\Delta H = 1/\beta$, curve 2: $\Delta H < 1/\beta$, curve 3: $\Delta H > 1/\beta$ (non-monotonic profile) and curve 4: $\Delta H = 0$.

The $H_x$ profile for three variants of the relation between $\Delta H$ and $\beta$, $\Delta H = \beta$, $\Delta H > \beta$ and $\Delta H < \beta$, is shown schematically in figure 6($a$).

### 5.1.3. *Small ordinary viscosity, $P_m \ll 1$: vortex sheet*

Here $\lambda_1 \approx \frac{1}{4}(1 + P_m\beta^2)$, $\lambda_2 \approx \frac{1}{4}P_m\mu$ and

$$\left.\begin{aligned} v_x &= U + \tfrac{1}{2}[1 + (1 - \beta\Delta H)\,\mathrm{erf}\,(\tfrac{1}{2}\zeta\sqrt{U}) + \beta\Delta H\,\mathrm{erf}\,(\tfrac{1}{2}\zeta\sqrt{P_m U\mu}\,)] \\ &\approx U + \tfrac{1}{2}[1 + (1 - \beta\Delta H)\,\mathrm{sign}\,(\zeta) + \beta\Delta H\,\mathrm{erf}\,(\tfrac{1}{2}\zeta\sqrt{P_m U\mu}\,)] \\ H_x &= H_- + \tfrac{1}{2}\Delta H[1 + \mathrm{erf}\,(\tfrac{1}{2}\zeta\sqrt{P_m U\mu}\,)] + O(P_m). \end{aligned}\right\} \tag{5.18}$$

In this case the situation with the scales is the opposite compared to the previous case: $\delta_W \sim (\nu_m x/U)^{1/2}$, $\delta_N \sim (\nu x/U)^{1/2}$. The magnetic field is determined only by its 'own' viscosity, i.e. varies mainly over a scale $\delta_W$, while the velocity profile has a two-scaled structure. It has near $\zeta = 0$ the 'jump' $v_x(+0) - v_x(-0) = 1 - \beta\Delta H$ of width $\delta_N \sim \delta_W P_m^{1/2}$ against the background of a smooth variation of magnetic field profile (a vortex sheet), but in the rest of the region it varies smoothly with the scale $\delta_W$. Note that the velocity profile is monotonic only if $\Delta H < 1/\beta$.

When $\Delta H = 0$ the magnetic field becomes uniform, the velocity profile contains only the narrow (based on usual viscosity) scale $\delta_N$ and in this case the full velocity jump $\Delta U = 1$ takes place in the narrow viscous region near $\zeta = 0$. This behaviour completely matches to the solution for an inviscid vortex sheet in the uniform parallel magnetic field which was described in § 3.2 (see (3.10) and (3.11)).

Note that only in the case of a uniform magnetic field is the viscous spreading of the flow the same as in the case without magnetic field. For the case with finite magnetic field difference the viscous spreading is faster.

The $v_x$ profile for four variants of the relation between $\Delta H$ and $1/\beta$ ($\Delta H = 1/\beta$, $\Delta H < 1/\beta$, $\Delta H > 1/\beta$ and $\Delta H = 0$) is schematically shown in figure 6($b$).

5.1.4. *Equal viscosities, $P_m = 1$*

We have $\lambda_1 = \frac{1}{4}(1+\beta)$, $\lambda_2 = \frac{1}{4}(1-\beta)$ and

$$\left.\begin{aligned} v_x &= U + \tfrac{1}{2}\{1 + \tfrac{1}{2}(1-\Delta H)\,\mathrm{erf}\,[\tfrac{1}{2}\zeta\sqrt{(1+\beta)U}\,] + \tfrac{1}{2}(1+\Delta H)\,\mathrm{erf}\,[\tfrac{1}{2}\zeta\sqrt{(1-\beta)U}]\},\\ H_x &= H_- + \tfrac{1}{2}\{\Delta H - \tfrac{1}{2}(1-\Delta H)\,\mathrm{erf}\,[\tfrac{1}{2}\zeta\sqrt{(1+\beta)U}] + \tfrac{1}{2}(1+\Delta H)\,\mathrm{erf}\,[\tfrac{1}{2}\zeta\sqrt{(1-\beta)U}]\}. \end{aligned}\right\} \tag{5.19}$$

A comparison of this solution with the exact solutions of § 4 shows that, in full accord with the results obtained in § 4, when $\Delta H = -1$ the solution (5.19) coincides with the exact narrow solution, and when $\Delta H = 1$, with the exact wide solution. Note, however, that the superposition (5.19) of the wide and narrow solutions satisfying the boundary conditions with an *arbitrary* given value of the difference $\Delta H$ is no longer an exact solution of the initial system (2.8), (2.9) at finite (not small) values of $U^{-1}$, because this system is nonlinear.

### 5.2. *Sub-Alfvén flows, $\beta > 1$*

When $\beta > 1$, the linearly independent solution $\Phi_W$ must be discarded because $\lambda_2 < 0$ (see (5.9)). Consequently, the constant $A_W$ in (5.11) must be set equal to zero. Yet this means that all necessary boundary conditions are no longer satisfied. In more exact terms the magnetic field difference $\Delta H$ when $\beta > 1$ cannot be specified arbitrarily. Indeed, from (5.10) and (5.12) it follows that the magnetic field and velocity differences when $A_W = 0$ must be related by

$$\Delta H \equiv \int_{-\infty}^{\infty} \Gamma_0 \mathrm{d}Z = \frac{1-4\lambda_1}{\beta}\int_{-\infty}^{\infty} \Phi_0 \mathrm{d}Z \equiv \frac{1-4\lambda_1}{\beta}\,\Delta U.$$

Hence we arrive at the conclusion that when $\beta > 1$ and at a given value of the difference $\Delta H$ other than $\Delta H_*$, where $\Delta H_* = (1-4\lambda_1)/\beta < 0$, or

$$\Delta H_* = \frac{1}{2\beta}\{(1-P_m) - [(1-P_m)^2 + 4P_m\beta^2]^{1/2}\}, \tag{5.20}$$

the stationary solution does not exist.

Here we are considering a very peculiar kind of situation. It turns out that the original problem, which is formulated as a boundary-value problem, becomes, when $\beta > 1$, the problem of seeking the eigenvalue whose role at a given $\beta$ is played by the magnetic field difference $\Delta H$.

Sub-Alfvén flow appears to be possible, not at an arbitrary value but at a particular value of $\Delta H$, $\Delta H = \Delta H_*$. The plausibility of such a conclusion is also confirmed indirectly by the reasoning that, as follows from (5.20), when $P_m = 0$ the 'eigenvalue' $\Delta H_* = 0$, and when $P_m = 1$ we have $\Delta H_* = -1$. This is consistent with exact results obtained in §§ 3 and 4 for these two values of $P_m$, for which sub-Alfvén flows, as is already known, do exist, and precisely at the above values of the differences $\Delta H$.

It can be shown, however, that this conclusion about the possible existence of sub-Alfvén solutions for arbitrary $P_m$ ($P_m \neq 0$, $P_m \neq 1$) and with the fixed magnetic field difference $\Delta H = \Delta H_*$ is not correct. The incorrectness of the above reasoning implies that it refers only to the zero-order approximation in $\varepsilon$. It turns out that even if $\Delta H = \Delta H_*$, and there is no growing solution $\Phi_W$ at the zero order, then at the next order in $\varepsilon$ the solution of the problem (5.5) is of necessity unbounded. This unboundedness cannot be eliminated by any refinement (of $O(\varepsilon)$) of the value of $\Delta H_*$. The unbounded solution does not arise only for two, already known, values of $P_m$,

$P_m = 0$ and $P_m = 1$, for which the exact sub-Alfvén solution exists at arbitrary $U$ (and hence the boundedness occurs in all orders of perturbation theory in $\varepsilon$).

The proof of the absence of bounded solutions of the first-approximation problem is given in the Appendix.

## 6. Discussion

*Summary*

The structure of the mixing layer between two parallel flows with different velocities and longitudinal magnetic fields is studied in the framework of a model of uniform incompressible fluid with constant magnetic and usual viscosity coefficients.

For the case when relative differences of the velocity $\Delta v_x$ and magnetic field $\Delta H_x$ across the layer are small, $\Delta v_x \ll \bar{v}_x$, $\Delta H_x \ll \bar{H}_x$ explicit analytical expressions for velocity and magnetic field profiles are obtained. The solutions involve two parameters – the magnetic Prandtl number $P_m$ and the interaction parameter $\beta = \bar{c}_A/\bar{v}_x$ – and demonstrate a two-scaled structure, which is described by superposition of two probability integrals. In the case of very different viscosities, velocity and magnetic field profiles contain regions of abrupt jumps against the background of a smooth variation, corresponding to either vortex sheet (when $P_m \ll 1$, or $Re \gg Re_m \gg 1$), or a $j_z$-electric current sheet (when $P_m \gg 1$, or $Re_m \gg Re \gg 1$) respectively. These solutions could be considered as a direct generalization of the solution for the so-called *Blasius* mixing layer (2.12), which is well-known in hydrodynamics.

It is shown that when the relation between the magnetic field and velocity differences is arbitrary, a stationary solution is possible only for super-Alfvén flows, $\beta < 1$. It is proved, however, that with a special choice of this relation sub-Alfvén mixing layers are also possible. But such a situation can be realized only for two values of the magnetic Prandtl number: $P_m = 0$ ($\nu = 0$) and $P_m = 1$ ($\nu = \nu_m$). For these two cases exact solutions are even obtained.

In the case of $P_m = 0$ the sub-Alfvén solution corresponds to the well-known vortex sheet in a uniform strongly parallel magnetic field, $\Delta H_x = 0$, $H_x = \text{const}$, $H_y = 0$.

In the case of $P_m = 1$ the sub-Alfvén solution corresponds to a flow in which the propagation rate of vorticity perturbations relative to a system of reference at rest is constant, $c_A + v_x (\equiv H_x + v_x) = \text{const}$. Such a flow can be realized only if the given magnetic field difference is equal in value and opposite in sign to the velocity difference, $\Delta H_x = -\Delta v_x$.

Apparently, both these sub-Alfvén solutions must be treated as degenerate. Indeed, the case when the viscosity is strictly zero cannot be treated as physically reasonable, but as we have shown in the Appendix, an arbitrarily small deviation of $P_m$ from zero leads to the impossibility of sub-Alfvén flow. The same holds true for the case of exact equality of viscosities. An arbitrarily small value of their difference, $|\nu - \nu_m| \ll \nu$, or violation of the condition $\Delta H_x = -\Delta v_x$ both lead to destruction of the stationary picture.

*Physical interpretation*

Below we shall try to give an interpretation of the absence of sub-Alfvén flows in the general case ($P_m \neq 1$) and their possibility in the special case $P_m = 1$ in terms of forces. (The reason for the possibility of sub-Alfvén flow with a vortex sheet in pure inviscid fluid, $P_m = 0$, in a strongly parallel uniform magnetic field is obvious and does not require comment, see also § 3.2.) The velocity (or vorticity) profile is determined by the balance of two effects: the advective transfer of the vorticity downstream, and viscous spreading in a transverse direction. The ponderomotive force prevents the advective transfer of the vorticity, and spreading effects begin to dominate, leading to a larger thickness of the layer than in the absence of the magnetic field. This is described by equation (2.3), which we now write in the form

$$(\boldsymbol{v}\cdot\nabla)v_x-(\boldsymbol{H}\cdot\nabla)\,H_x=\nu\frac{\partial^2 v_x}{\partial y^2}. \tag{6.1}$$

In the case of a sufficiently large magnetic field $H_x \approx v_x$ the mixing layer spreads to infinity. However, in the case $P_m = 1$, and, in addition, with a special choice of the boundary conditions, i.e. at $\Delta H_x = -\Delta v_x$, a sub-Alfvén flow is also realized. Let the gradients of magnetic field and velocity components at each point of the flow be equal in value and opposite in sign, $\nabla H_x = -\nabla v_x$, $\nabla H_y = -\nabla v_y$, i.e.

$$H_x=-v_x+c_1,\quad H_y=-v_y,\quad c_1=\text{const}. \tag{6.2}$$

One can understand, using the induction equation (2.4), that with this relationship between magnetic field and velocity differences, as well as due to the equality of rates of diffusive spreading ($\nu_m = \nu$), such a relationship between the velocity and the magnetic field is indeed possible. But in such a case from (6.2) and (6.1) it follows that

$$\frac{\partial\Omega}{\partial t}+c_1\frac{\partial\Omega}{\partial x}=\nu\,\frac{\partial^2\Omega}{\partial y^2}. \tag{6.3}$$

Here $\Omega = \partial v_x/\partial y$ is the vorticity, and for illustrative purposes we added to the left-hand side of (6.3) a non-stationary term (equal to zero in the steady-state problem under consideration). Hence it turns out that with such a special *self-sustaining* relationship between the magnetic field and the velocity, the inertial force everywhere exceeds the electromagnetic force at any magnitude of the magnetic field, including the case of $c_A > v_x$, and the vorticity is carried with a constant velocity $c_1 = v_x(x,y)+c_A(x,y)$. Equation (6.3) represents the usual diffusion equation with a constant positive diffusion coefficient $D = \nu/(v_x + c_A)$. Note that the solution just described is a particular case of the *narrow* solution ($c_1 = 4\lambda_1$), corresponding to choice $\xi_0 = 0$, when $v_y(\pm\infty) = H_y(\pm\infty) = 0$ (see § 4 and figures 3 and 4).

It can be said that in this special case it is possible to extract a ‘good’ solution corresponding to the Alfvén wave propagating downstream (that is, the solution of equation (6.3)) from the ‘bad’ solution corresponding to the propagation of the vorticity with the velocity $c_2 = v_x - c_A$. This velocity is directed upstream in the sub-Alfvenic case $c_A > v_x$, and the ‘impurity’ of such a solution destroys the stationary picture. When $P_m \neq 1$, as well as when $P_m = 1$, but $\Delta H_x \neq -\Delta v_x$, such a splitting is impossible, which leads to the impossibility of sub-Alfvén flows.

Note that the specificity of the value $P_m = 1$ for MHD was indicated by Hasimoto (1959*b*) who has shown that for the case $P_m = 1$ the MHD equations have some interesting properties, and, in particular, permit exact nonlinear solution in a form of dissipative waves.

### *Remarks on the impossibility of partially sub-Alfvén flow*

Note that, strictly speaking, the conclusion about the impossibility of sub-Alfvén flows has thus far been drawn only on the basis of results relevant to flows with small relative differences. However, the validity of this conclusion for arbitrary flows is virtually beyond question. An analysis of the asymptotic behaviour of the solutions of the system (2.8), (2.9) as $\xi \to \pm\infty$ also confirms such a conclusion. Indeed, for the asymptotic behaviour of two independent solutions constituting $f''$ and $\chi''$, we have $\Phi^{\pm}_{N,W}(\xi) \propto \exp\,[-\lambda^{\pm}_{N,W} U_{\pm}(\xi - \xi^{\pm}_0)^2]$, where

$$\lambda^{\pm}_N = \tfrac{1}{8}(1 + P_m)\,(1 + Q_{\pm}), \quad \lambda^{\pm}_W = \tfrac{1}{8}(1 + P_m)\,(1 - Q_{\pm}),$$
$$Q_{\pm} = [1 + 4P_m\,(\beta^2_{\pm} - 1)/(1 + P_m)^2]^{1/2},$$

and $\beta_{\pm} = H_{\pm}/U_{\pm}$, $U_- = U$, $U_+ = U + 1$. Here the + and − signs refer to the asymptotic behaviour as $\xi \to +\infty$ and $\xi \to -\infty$ respectively, and in the general case $\xi^+_0 \neq \xi^-_0$.

Hence it follows that if at least one of $\beta_{\pm}$ tends to unity, the 'half-thickness' of the layer on the respective side tends to infinity. It is also evident that as long as $\beta_{\pm} < 1$, we can find (only numerically) the solution satisfying all boundary conditions imposed.

If, however, at least on one side the flow becomes sub-Alfvén, i.e. at least one of $\beta_+$ or $\beta_-$ becomes larger than unity, the corresponding value of $\lambda_W$ becomes negative. In this case $\Phi_W$ increases with no limit in the respective direction, and the boundedness condition of $f''$ and $\chi''$ no longer holds automatically. Hence, with arbitrary imposed boundary conditions at which at least on one side $\beta > 1$, there is certainly no solution. In principle, it is still not inconceivable that the boundedness condition can be satisfied by fitting some of the boundary values of the magnetic field (at fixed boundary values of the velocity). Most likely, however, such a possibility (except for $P_m = 0$ and $P_m = 1$) also cannot be realized. In the case with small relative differences we have demonstrated this rigorously (in the Appendix). In the case of arbitrary, not small, differences, however, our attempt to determine numerically the 'eigenvalue' of $H_+$ for given $H_-$ and $U$, has not met with success.

Thus it appears that the conclusion about the impossibility of at least a partially sub-Alfvén mixing layer is quite convincing.

Note that a similar conclusion about impossibility of a stationary flow when the parameter of interaction $\beta = c_A/U$ exceeds unity has been made by Meksyn (1962) for the problem of the *boundary layer* on a flat plate. (Here $c_A$ and $U$ are Alfvén and flow velocities in the outer flow; on the plate itself magnetic field and velocity are absent.) However, it is interesting that even in the special case $P_m = 1$ the solution for $\beta > 1$ does not exist for a boundarys layers. The reason for such difference between mixing layers and boundary layers lies in the fact that in the case of a boundary layer the flow with $v_x + c_A = \text{const}$ cannot be created.

### *Results in the context of nonlinear evolution*

In closing it may be said that the results obtained here help to resolve a difficulty (or paradox) that emerged in Paper I (Shukhman 1998*a*) devoted to the study of the downstream nonlinear evolution of unstable perturbations excited by a periodic source in the mixing layer with a longitudinal magnetic field. This difficulty arose when the velocity profile included a region where the Alfvén velocity exceeded the fluid velocity. Formally, it implied that the equation for the zeroth harmonic of the perturbation (equation (A8) of Paper I), having the form of a diffusion equation with a source, lost its physical meaning: the downstream coordinate could no longer be treated as a time-like one, because the 'time arrow' in this case is directed backwards or, equivalently, the viscosity is negative. This means the violation of 'causality': regions lying downstream can influence those lying upstream. Therefore, to avoid this difficulty we restricted ourselves in Paper I only to the case of 'fast' (i.e. super-Alfvén) flows in which $u_{\min} > c_A$. In the light of the results reported in this paper, it becomes clear that this restriction is in fact a natural requirement which has to be imposed on background flow. In other words, the difficulty mentioned above was illusiory. It was associated with an incorrect attempt to describe the evolution of perturbations on the background of a sub-Alfvén mean flow which, as is now apparent, is impossible to create.

I am grateful to Dr S. M. Churilov for helpful discussions and Mr V. G. Mikhalkovsky for his assistance in preparing the English version of the text.

## Appendix. Proof of the impossibility of a bounded solution of the first order-approximation problem at $\beta > 1$

Denote $\tilde{F}_1'' = \Phi_1$, $\tilde{G}_1'' = \Gamma_1$. Then from (5.5) we obtain

$$\Phi_1' - \tfrac{1}{2}Z(\beta\Gamma_1 - \Phi_1) = \tfrac{1}{2}\varrho_1, \quad \frac{\Gamma_1'}{P_m} - \tfrac{1}{2}Z(\beta\Phi_1 - \Gamma_1) = \frac{\varrho_2}{\beta}, \tag{A 1}$$

$$\varrho_1 \equiv \tilde{G}_0\Gamma_0 - \tilde{F}_0\Phi_0, \quad \varrho_2 \equiv \tfrac{1}{2}\beta(\tilde{G}_0\Phi_0 - \tilde{F}_0\Gamma_0). \tag{A 2}$$

It is easy to reduce the system (A1) to a single equation for $\Phi_1$:

$$\mathscr{L}\Phi_1(Z) = ZR(Z), \quad \text{where} \quad R(z) \equiv \tfrac{1}{2}[\tfrac{1}{2}P_m + (\mathrm{d}/\mathrm{d}z)(z^{-1})]\varrho_1(z) + \tfrac{1}{2}P_m\varrho_2(z), \tag{A 3}$$

and the operator $\mathscr{L}$ is defined by (5.8). Recall that the solutions of the homogeneous equation $\mathscr{L}\Phi = 0$ are the functions $\Phi_N(z) = \mathrm{e}^{-\lambda_1 z^2}$ and

$$\Phi_W(z) = \mathrm{e}^{-\lambda_2 z^2} = \Phi_N(z)\left[1 - \int_0^z \mathrm{d}x\, W(x)\,\Phi_N^{-2}(x)\right], \tag{A 4}$$

where $\lambda_1 > 0$, $\lambda_2 < 0$. Following the general rules governing the solution of the inhomogeneous equation, the solution (A3) may be written as

$$\Phi_1(Z) = \Phi_N(Z)\int^Z \frac{zR(z)}{W(z)}\,\Phi_W(z)\,\mathrm{d}z - \Phi_W(Z)\int^Z \frac{zR(z)}{W(z)}\,\Phi_N(z)\,\mathrm{d}z,$$

or, using (A4), in the form

$$\Phi_1(Z) = \Phi_N(Z)\int_0^Z \frac{\mathrm{d}x\, W(x)}{\Phi_N^2(x)}\left[b_W + \int_{-\infty}^x \frac{z\Phi_N(z)}{W(z)}\,R(z)\,\mathrm{d}z\right] + b_N\Phi_N(Z). \tag{A 5}$$

We now choose a constant $b_W$ equal to zero. This means that $\Phi_1$ does not contain growing exponential $\sim \exp(-\lambda_2 Z^2)$ as $Z \to -\infty$. In order for $\Phi_1$ not to contain it also as $Z \to +\infty$ it is necessary to require that $I = \int_{-\infty}^{\infty}[z\Phi_N(z)R(z)/W(z)]\mathrm{d}z = 0$. This is, in principle, just the boundedness condition of the first-approximation solution. This involves some subtlety, however. The point is that, according to (A3), $R(z)$ has a second-order pole at $z = 0$, and $z\Phi_N/W$ is regular at $z = 0$, so that the integrand in $I$ also has a second-order pole. Therefore, before proceeding to verifying the boundedness condition, we transform (A5) and reformulate this condition in a form free from the above-mentioned difficulty with the divergence. On substituting $R(z)$ from (A3) into (A5) and integrating the singular term $(\mathrm{d}/\mathrm{d}z)(\varrho_1/z)$ by parts, we obtain

$$\Phi_1(Z) = \Phi_N(Z)\left[ba_N + \tfrac{1}{2}\int_0^Z \frac{\mathrm{d}x\,\varrho_1(x)}{\Phi_N(x)} - \int_0^Z \frac{\mathrm{d}x\, W(x)}{\Phi_N^2(x)}\,\Pi(x)\right], \tag{A 6}$$

where

$$\Pi(x) = \int_{-\infty}^{x} \mathrm{d}z \, \frac{z\Phi_N(z)}{W(z)} \left[ (\tfrac{1}{4} - \lambda_1)\varrho_1(z) - \tfrac{1}{2} P_m \varrho_2(z) \right]. \tag{A 7}$$

The boundedness condition of $\Phi_1$, as is evident from (A6), can now be formulated as $\Pi(+\infty) = 0$. The integrand in (A7) now contains no singularity at $z = 0$, and the integral $\Pi(\infty)$ is readily evaluated. From (5.11)–(5.14) we have, when $A_W = 0$,

$$\left.\begin{array}{l} \Phi_0 = (\lambda_1/\pi)^{1/2} \mathrm{e}^{-\lambda_1 z^2}, \quad \Gamma_0 = \Delta H_* \, \Phi_0, \\ \tilde{F}_0 = \tfrac{1}{2}\{[1 + \mathrm{erf}\,(z\sqrt{\lambda_1})]\, z + (\lambda_1 \pi)^{-1/2} \, \mathrm{e}^{-\lambda_1 z^2}\}, \quad \tilde{G}_0 = \Delta H_* \, \tilde{F}_0. \end{array}\right\} \tag{A 8}$$

Substituting (A8) into (A2) gives $\varrho_2 = 0$ and $\varrho_1 = [(\Delta H_*)^2 - 1]\tilde{F}_0 \Phi_0$. On performing the integration, we eventually obtain the expression for $\Pi(\infty)$:

$$\Pi(\infty) = \frac{1 - P_m}{4P_m} \frac{\lambda_1}{(\lambda_1 - \lambda_2)^2} \left( \frac{2\lambda_1 - \lambda_2}{\pi} \right)^{1/2} (\Delta H_*)^2. \tag{A 9}$$

The expression (A9) shows that the boundedness condition for $\Phi_1$, i.e. $\Pi(\infty) = 0$, *does not hold.*

The sole exception is the case $P_m = 1$. It can be shown that $\Pi(\infty)$ also becomes zero when $P_m = 0$. Indeed, when $P_m \to 0$ we have $\lambda_1 \approx \frac{1}{4}$, $\lambda_2 = O(P_m)$, and it follows from (5.20) that $\Delta H_* \approx -P_m \beta^2$. Therefore $\Pi(\infty) = (2\pi)^{-1/2} \beta^2 P_m \to 0$ when $P_m \to 0$.

Thus we have shown that sub-Alfvén ($\beta > 1$) flows with $P_m \neq 0$ and $P_m \neq 1$ cannot exist not only at arbitrary specified values of the magnetic field difference $\Delta H$ but also at $\Delta H = \Delta H_*$.

It should be noted in conclusion that for super-Alfvén flows, $\beta < 1$ , the transition to a first approximation does not give any additional limitations on the zero-approximation solution.

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